Optimality of the honeycomb structure in some shape optimization problems with inradius constraint in 2D

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Shape Optimization, Geometric Inequalities and Related Topics
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Definitions

Dirichlet eigenvalue for the p-Laplacian

Principal eigenvalue of the p-Laplacian of Ω with Dirichlet boundary conditons :

$$\lambda_p(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \;\middle|\; u \in W_0^{1,p}(\Omega) \right\} = \int_\Omega |\nabla u_p|^p,$$

where u_p is the associated eigenfunction and is the unique solution to

$$\begin{cases} -\Delta_p u_p = \lambda_p(\Omega) u_p^{p-1} \text{ in } \Omega, \\ u_p \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p = 1. \end{cases}$$

Definitions

Torsionnal rigidity for the p-Laplacian

p-Torsional rigidity of Ω,

$$T_p(\Omega) = \sup\left\{ \frac{\left(\int_\Omega |u|\right)^{\frac{p}{p-1}}}{\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p-1}}} \;\middle|\; u \in W_0^{1,p}(\Omega) \right\} = \int_\Omega w_p,$$

where w_p is the torsion function and is the unique solution to

$$\begin{cases} -\Delta_p w_p = 1 \text{ in } \Omega, \\ w_p \in W_0^{1,p}(\Omega). \end{cases}$$

Main problem

We work in $\mathbb{R}^2$, and we are interested in the study of problems of the form:

$$\min\left\{J(\Omega) \mid \Omega \in \mathcal{A},\ \rho(\Omega) = 1\right\},$$

with $\rho(\Omega)$ the inradius of Ω,
J can be :

- $\lambda_p(\Omega)$
- $\frac{|\Omega|}{T_p(\Omega)}$

Main problem

What is $\mathcal{A}$?

- $p > 2$,
$$\mathcal{A} = \left\{\Omega \subset \mathbb{R}^2,\ \Omega \text{ open},\ |\Omega| < +\infty\right\}$$
- $p \leq 2$, in that case points have zero capacity, so we must remove small holes
$$\mathcal{A} = \mathcal{A}_\varepsilon = \left\{\Omega = \mathbb{R}^2 \setminus \bigcup \overline{B(x_i, \varepsilon)} \;\middle|\; (x_i) \in \mathbb{R}^{\mathbb{N}}\right\},$$

Motivations

The torsion problem

- case $p = +\infty$

Theorem (Briani, Bucur - 2023)

For all Ω *open subset of* $\mathbb{R}^2$ *of finite measure, and inradius* 1

$$\frac{T_\infty(\Omega)}{|\Omega|} = \frac{\int_\Omega d(x, \partial\Omega)}{|\Omega|} \leq \lim_{N \to +\infty} \frac{T_\infty(B(0, N) \cap (\mathbb{R}^2 \setminus Z_{hex}))}{\pi \ N^2}$$

where Z_{hex} *is the set of the centers of the hexagonal tiling of inradius* 1.

- case $p = 2$, the question has already arised:
  - 2002 - Morgan and Bolton, for problems comming from economics
  - 2006 - Buttazzo, Santambrogio and Varchon

In both cases, the conjecture was that the hexagonal tiling should be the optimal distribution.

Motivation

The eigenvalue problem - a small digression

Considering the functionnal, for $1 < p < q \leq +\infty$,

$$\mathcal{F}_{p,q}(\Omega) = \frac{\lambda_q(\Omega)^{1/q}}{\lambda_p(\Omega)^{1/p}},$$

we have the existence result

Theorem (Bucur, Buttazzo, V - 2025)

For any $2 < p < q \leq +\infty$, $\mathcal{F}_{p,q}$ admits a maximizer of the form $\Omega_{opt} = \mathbb{R}^2 \setminus Z$ where Z is a discrete set with no accumulation points

Motivation

The eigenvalue problem - link with the inradius

We know $\lambda_q(\Omega)^{1/q} \to 1/\rho(\Omega)$, therefore

$$\mathcal{F}_{p,\infty} = \frac{1}{\lambda_p(\Omega)^{1/p}\ \rho(\Omega)}$$

then, by scale invariance,

$$\max \mathcal{F}_{p,\infty}(\Omega) \iff \min \left\{ \lambda_p(\Omega) \middle|\ \rho(\Omega) = 1 \right\}$$

Proof of the existence result with drawings

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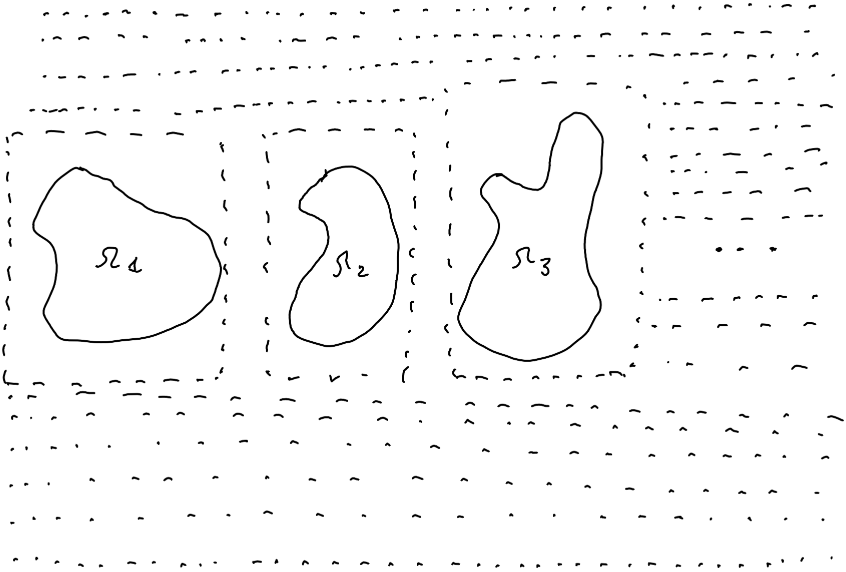
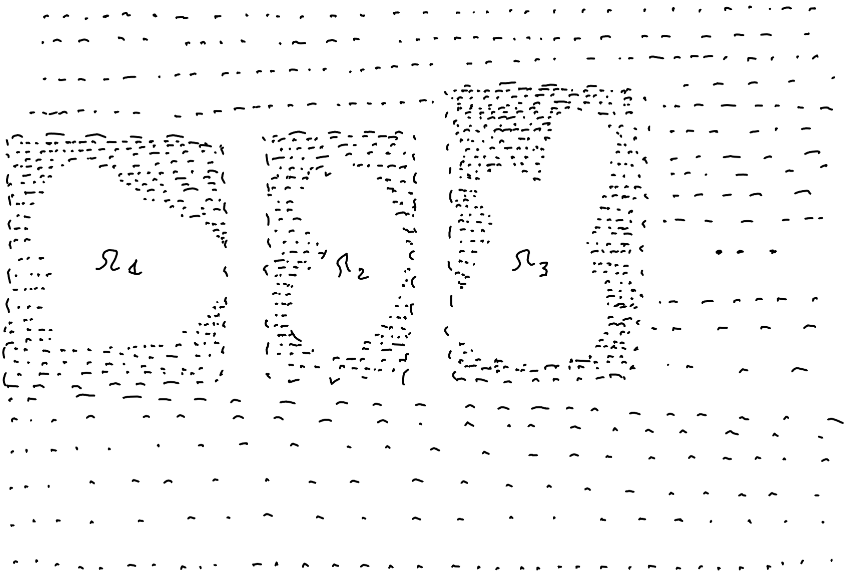


Proof of the existence result with drawings



Main question

Question : Can we prove that for any $\Omega \in \mathcal{A}$ of inradius 1,

- $\lambda_p(\Omega) \geq \lambda_p(\mathbb{R}^2 \setminus Z_{hex} \oplus \overline{B(0,\varepsilon)})$,
- $\frac{T_p(\Omega)}{|\Omega|} \leq \lim_{N\to+\infty} \frac{T_\infty(B(0,N)\cap(\mathbb{R}^2 \setminus Z_{hex} \oplus \overline{B(0,\varepsilon)}))}{\pi \ N^2}$?

Answer : Not really, but we have some ideas

General strategy

- (easy) Reduction to a problem on triangles

$$\min\{J(\Omega)|\ \Omega \in \mathcal{A},\ \rho(\Omega) = 1\} \geq \min\{J^*(\triangle)|\ R(\triangle) = 1\}$$

  - consider $\Omega = \mathbb{R}^2 \setminus Z \oplus \overline{B(0, \varepsilon)}$
  - take a Delauney triangulation $\mathcal{F}$ of the discrete set Z,
  - show $J(\Omega) \geq \min_{\triangle \in \mathcal{F}} J^*(\triangle)$ with equality when $Z = Z_{hex}$
- (hard) Solve the problem on triangles and show

$$\min\{J^*(\triangle)|\ R(\triangle) = 1\} = J^*(\triangle_{eq})$$

What is J^* ?

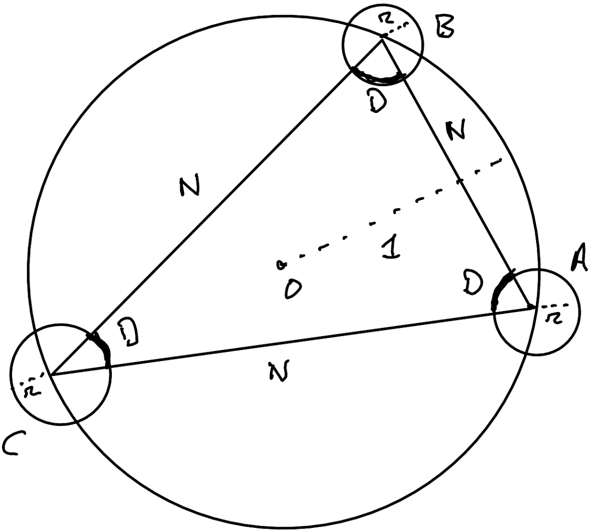
For $p > 2$, we define for any triangle $\triangle$ of vertices A_1, A_2, A_3,

- $\lambda_p^*(\triangle) = \inf \left\{ \frac{\int_\triangle |\nabla u|^p}{\int_\triangle |u|^p} \;\middle|\; u \in W^{1,p}(\triangle),\ \forall i,\ u(A_i) = 0 \right\}$,
- $T_p^*(\triangle) = \sup \left\{ \frac{\left(\int_\triangle |u|\right)^{\frac{p}{p-1}}}{\left(\int_\triangle |\nabla u|^p\right)^{\frac{1}{p-1}}} \;\middle|\; u \in W^{1,p}(\triangle),\ \forall i,\ u(A_i) = 0 \right\}$.

For $p \leq 2$ it is the same but with Dirchlet boundary conditions on the small balls of radius r at each vertex.

Reduction step

The drawing for the problems on triangles



Reduction step

Computation for the Eigenvalue (not rigourous)

Let $\Omega = \mathbb{R}^2 \setminus Z$ be a competitor for λ_p, with Z discrete with no acumulation points. Define $\mathcal{F}$ a Delauney triangulation of Z, then

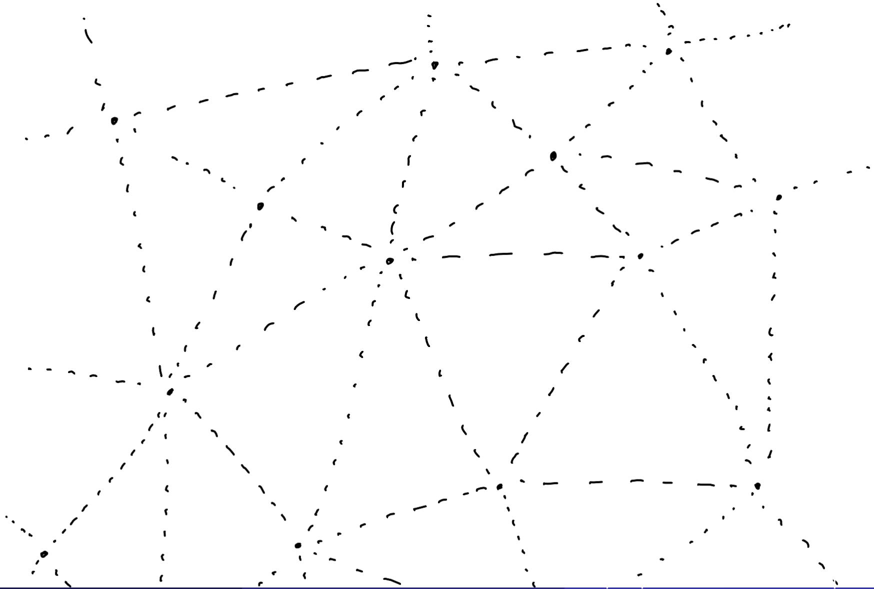
$$\begin{aligned}\lambda_p(\Omega) &= \frac{\sum_{\triangle\in\mathcal{F}} \int_\triangle |\nabla u_p|^p}{\sum_{\triangle\in\mathcal{F}} \int_\triangle |u_p|^p} \\ &\geq \inf_{\triangle\in\mathcal{F}} \frac{\int_\triangle |\nabla u_p|^p}{\int_\triangle |u_p|^p} \\ &\geq \inf_{\triangle\in\mathcal{F}} \lambda_p^*(\triangle) \\ &\geq \inf \left\{ \lambda_p^*(\triangle) \;\middle|\; R(\triangle) = 1 \right\}\end{aligned}$$

Reduction step

The drawing for the triangulation

Reduction step

The drawing for the triangulation



The results

$p = 2$, asymptotic behavior for the Torsion problem

We show the following for the torsion problem

Theorem (Bucur, Buttazzo, V)

For any acute or right triangle $\triangle$, of circumradius $R(T) = 1$, it holds

$$\lim_{r\to 0} \frac{1}{|\ln(r)|} \frac{T_r^*(\triangle)}{|\triangle| - \pi/2\ r^2} = \frac{|\triangle|}{\pi}.$$

And as a corollary, **with an additional symmetry argument to rule out obtuse triangles**, we obtain that in the limit the quantity,

$$\frac{1}{|\ln(r)|} \frac{T_r^*(\triangle)}{|\triangle| - \pi/2\ r^2},$$

is maximal for the equilateral triangle.

The results

$p = 2$, asymptotic behavior for the eigenvalue problem

We show

Theorem (Bucur, Buttazzo, V)

For any triangle $\triangle$, it holds

$$\lim_{r \to 0} \lambda_r^*(\triangle)\ T_r^*(\triangle) = |\triangle|.$$

And therefore, we obtain that the quantity,

$$\ln(r)\ \lambda_r^*(\triangle),$$

is minimal in the limit for the equilateral triangle

The results

$p = 1$, the Cheeger case

For any open set Ω in $\mathbb{R}^2$, we define the Cheeger constant of Ω, as

$$h(\Omega) = \inf \left\{ \frac{p(E)}{|E|} \;\middle|\; E \Subset \Omega \right\}.$$

It is possible to show that actually

$$h(\Omega) = \lambda_1(\Omega).$$

The adapted cheeger constant becomes

$$h_r^*(\triangle) = \inf \left\{ \frac{p(E, \triangle)}{|E|} \;\middle|\; E \subset \triangle \setminus \bigcup_i B(A_i, r) \right\}.$$

The results

$p = 1$, the Cheeger case

Theorem (Butcur, Buttazzo, V)

There exists an $r^* > 0$ (explicit) such that for any $r < r^*$, it holds for any triangle $\triangle$ of circumradius $R(\triangle) = 1$,

$$h_r^*(\triangle) \geq h_r^*(\triangle_{eq}),$$

where $\triangle_{eq}$ is the equilateral triangle of circumradius 1.

As a corollary, we obtain the result in the whole space

Theorem

There exists ε^* sufficiently small (explicit) such that for any $\varepsilon < \varepsilon^*$, it holds for any set $\Omega \in \mathcal{A}_\varepsilon$, of inradius $\rho(\Omega) = 1$,

$$h(\Omega) \geq h\left(\mathbb{R}^2 \setminus \left((1+\varepsilon) Z_{hex} \oplus \overline{B(0,\varepsilon)}\right)\right)$$

What's next ?

At the moment we work on showing

Theorem (Hopefully true)

There exists $r^* > 0$*, such that if* $r < r^*$*, then for all triangle* $\triangle$ *of circumradius* $R(\triangle) = 1$,

$$\frac{T_r^*(\triangle)}{|\triangle| - \frac{\pi\ r^2}{2}} \leq \frac{T_r^*(\triangle_{eq})}{|\triangle_{eq}| - \frac{\pi\ r^2}{2}}$$

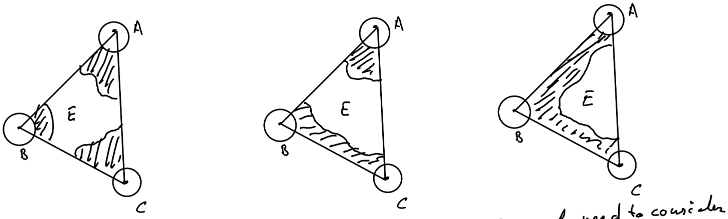
And then ?

- Try to use the limit of $\lambda_r^*(\triangle) T_r^*(\triangle)$ to obtain the same behavior for $\lambda_r^*(\triangle)$
- Try to understand the behavior for $T_p^*(\triangle)$ when p is close to $+\infty$
- ... solve the full problem ...

Thank you very much for your attention !!

Bonus

"Proof" of the Cheeger case



Isoperimetric inequality

$$\frac{P(E,\Delta)}{|E|} \geq \frac{\pi r}{|\Delta| - \frac{\pi r^2}{2}}$$

$$\geq \frac{\pi r}{|\Delta_{eq}| - \frac{\pi r^2}{2}}$$

$\vdots$

$$\frac{P(E,\Delta)}{|E|} \geq 1.$$

r small

$\downarrow$

[you only need to consider

$p = p(E) < \pi r$]

and, isoperimetric inequality (half disk)

$$|E| < \frac{p^2}{2\pi}$$

$$\frac{P(E,\Delta)}{|E|} \geq \frac{p}{\frac{p^2}{2\pi}} = \frac{2\pi}{p}$$

$$\geq \frac{2}{r}$$

$$\Rightarrow \quad h_r(T) \geq \min\left(\frac{\pi r}{|\Delta_{eq}| - \frac{\pi r^2}{2}},\ 1,\ \frac{2}{r}\right) = \frac{\pi r}{|\Delta_{eq}| - \frac{\pi r^2}{2}} = h_r(\Delta_{eq})$$

Bonus 2

Where are the hexagons ?