Geometric shape optimization and honeycomb structure

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What is a shape optimization problem ?

We consider the problem

$$\min \left\{ F(\Omega) \;\middle|\; \Omega \in \mathcal{A} \right\}.$$

Here $\mathcal{A}$ is a class of subsets of $\mathbb{R}^d$ and is called the class of admissible shapes

Typically:

- Open subsets of $\mathbb{R}^d$
- Open subsets of $D \subset \mathbb{R}^d$
- Convex subsets of $\mathbb{R}^d$

Usual constraints:

- fixed volume
- fixed perimeter
- fixed inradius

Isoperimetric inequality

The setting

- $\Omega \subset \mathbb{R}^d$, open
- $|\Omega|$ measure of Ω
- $p(\Omega)$ perimeter of Ω
- B_Ω ball of measure $|\Omega|$

$$p(\Omega) \geq p(B_\Omega)$$

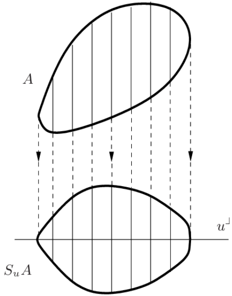
Isoperimetric inequality

Imcomplete proofs 19th century

Theorem (Steiner 1860s)

If the set Ω (is smooth and) is **not** *the ball there exists a strictly better set* Ω^*

Steiner symmetrization:



This is not a complete proof :

Full proof : De Giorgi 1954

Isoperimetric inequality

Imcomplete proofs 19th century

Theorem (Steiner 1860s)

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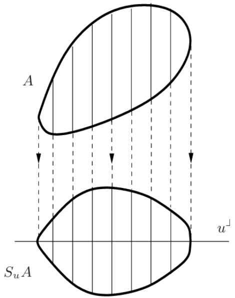


This is not a complete proof :

$$\forall n \in \mathbb{N},\ n \neq 1,\ n < n^2 \not\Rightarrow 1 = \max \mathbb{N}$$

Full proof : De Giorgi 1954

The honeycomb structure

The historic problem

Theorem (Hales - 1999)

Among all partitions of the plane in cells of equal area the hexagonal tesselation (or honeycomb structure) has the least average perimeter

Very old problem:

- 36 BC Marcus Terentius Varro
- 5th century Pappus of Alexandria
- 1943 L Fejes Tóth - convex case

The honeycomb structure

the modern problems

- 2002 - Morgan and Bolton : Distribution of resources in an economic framework
- 2006 - Buttazzo, Santambrogio and Varchon : Optimal compliance location problem
- 2007 - Caffarelli and Lin : Optimal partition probem for eigenvalues
- 2023 - Briani Bucur : Optimality of the ∞-torsion under inradius constraint

Faber Krahn inequality

Defnitions and old result

Principal eigenvalue of the Dirichlet Laplacian:

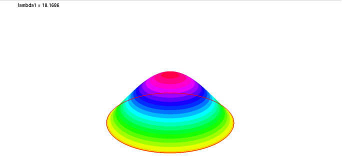
$$\lambda(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$$

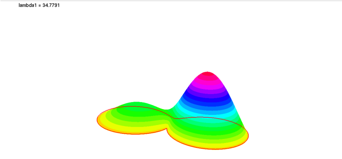
Some properties:

- if $\Omega' \subset \Omega$, then $\lambda(\Omega') \geq \lambda(\Omega)$
- for all $t > 0$, $\lambda(t\Omega) = t^{-2}\lambda(\Omega)$

Faber Krahn inequality (1923) :

$$\lambda(\Omega) \geq \lambda(B_\Omega)$$





Faber Krahn inequality

New results : the case of polygons

Questions :

- Is it true that among all n-gons, the regular n-gon P_n minimizes λ ?
- Is it true that $\lambda(P_n)$ is decreasing ?
- Is the curve convex ?

Answers :

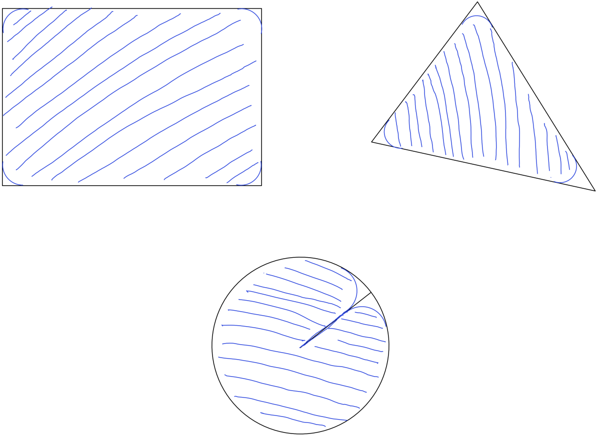
- Bogosel, Bucur 2022 2024 : local minimality seems true for $n = 7, 8$ and holds for $n = 5, 6$
- Dahne, Gomez Serrano, Pech-Alberich 2026 : true
- still open

Cheeger constant

Definitions

$\Omega \subset \mathbb{R}^d$,

$$h(\Omega) = \inf \left\{ \frac{p(E)}{|E|} \;\middle|\; E \Subset \Omega \right\}$$



Cheeger constant

Properties

Basic properties :

- if $\Omega' \subset \Omega$, then $h(\Omega') \geq h(\Omega)$
- for all $t > 0$, $h(t\Omega) = t^{-1}h(\Omega)$

Link with the eigenvalue problem :

- Alternative definition

$$h(\Omega) = \inf_{u \in W^{1,1}(\Omega)} \frac{\int_\Omega |\nabla u|}{\int_\Omega |u|}$$

- Faber Krahn inequality

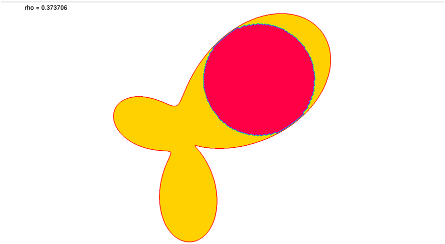
$$h(\Omega) \geq h(B_\Omega)$$

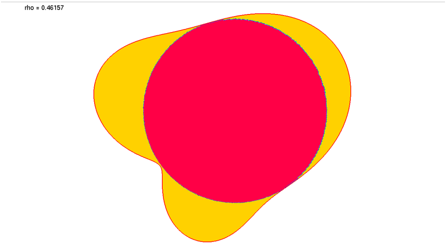
Inradius

Definition

$\Omega \subset \mathbb{R}^d$,

$$\rho(\Omega) = \inf \left\{ r \;\middle|\; \exists x \in \Omega,\ B(x,r) \subset \Omega \right\}.$$





Optimization of the Cheeger constant with fixed inradius

not very interesting ...

Problems we look at :

1. $S = \sup h(\Omega)$ for $\rho(\Omega) = 1$
2. $I = \inf h(\Omega)$ for $\rho(\Omega) = 1$

Solutions are trivial :

1. $S = h(B^*)$
2. $I = 0$ with $\Omega^* = \mathbb{R}^2 \setminus \xi\mathbb{Z}^2$

Optimization of the Cheeger constant with fixed inradius

Modified setting

We introduce

$$\mathcal{A}_\varepsilon = \left\{ \Omega = \mathbb{R}^2 \setminus (Z \oplus B(0,\varepsilon)) \;\middle|\; Z \subset \mathbb{R}^2,\ \text{discrete} \right\}$$

And consider

$$\inf \left\{ h(\Omega) \;\middle|\; \Omega \in \mathcal{A}_\varepsilon,\ \rho(\Omega) = 1 \right\}$$

Optimization of the Cheeger constant with fixed inradius

much more interesting ...

Theorem (Bucur, Buttazzo, AdV)

Let $\Omega^* = \mathbb{R}^2 \setminus ((1+\varepsilon)Z_{hex} \oplus B(0,\varepsilon))$*, then there exists* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon < \varepsilon^*$ *and* $\Omega \in \mathcal{A}_\varepsilon$ *satisfying* $\rho(\Omega) = 1$*,*

$$h(\Omega) \geq h(\Omega^*)$$

where Z_{hex} is the set of the centers of the hexagonal tiling of radius 1,

$$Z_{\text{hex}} = \left\{ \sqrt{3}n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{3}m \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \;\middle|\; (n,m) \in \mathbb{Z}^2 \right\}$$

Optimization of the Cheeger constant with fixed inradius

Idea of the proof

1. Reduction step
   Use a Delauney triangulation and show

$$h(\Omega) \geq \frac{1}{1+\varepsilon} \inf h^*_{r_\varepsilon}(\triangle)$$

   for $\triangle$ with circumradius $R(\triangle) = 1$

2. Solve the reduced problem
   For all $\triangle$ such that $R(\triangle) = 1$, and r small

$$h^*_r(\triangle) \geq h^*_r(\triangle_{\text{eq}})$$

reduced Cheeger constant

Who is h_r^* ?

For a triangle $\triangle$ of vertices A_i,

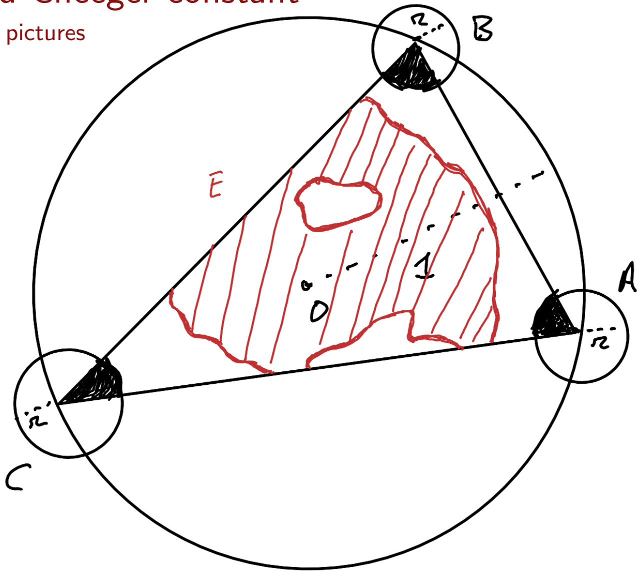
$$h_r^*(\triangle) = \inf \left\{ \frac{p(E,\triangle)}{|E|} \;\middle|\; E \subset \triangle \setminus \cup_i B(A_i, r) \right\}$$

and in analytic formulation

$$h_r^*(\triangle) = \inf \left\{ \frac{\int_\triangle |\nabla u|}{\int_\triangle |u|} \;\middle|\; u \in W^{1,1}(\triangle),\ u = 0 \text{ on } \cup_i B(A_i, r) \right\}$$

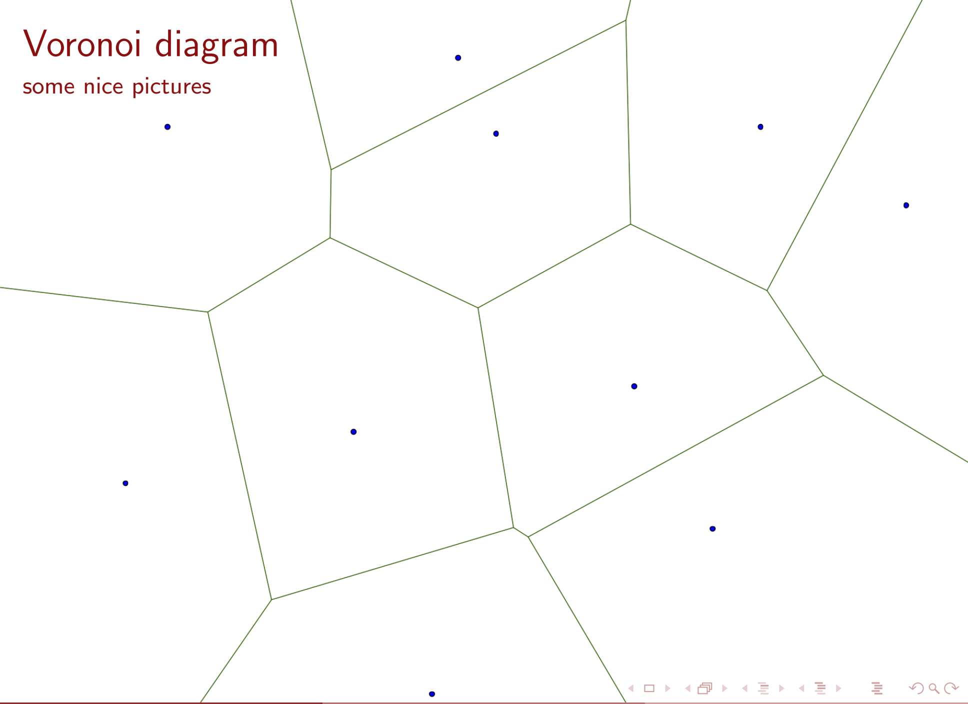
reduced Cheeger constant

some nice pictures



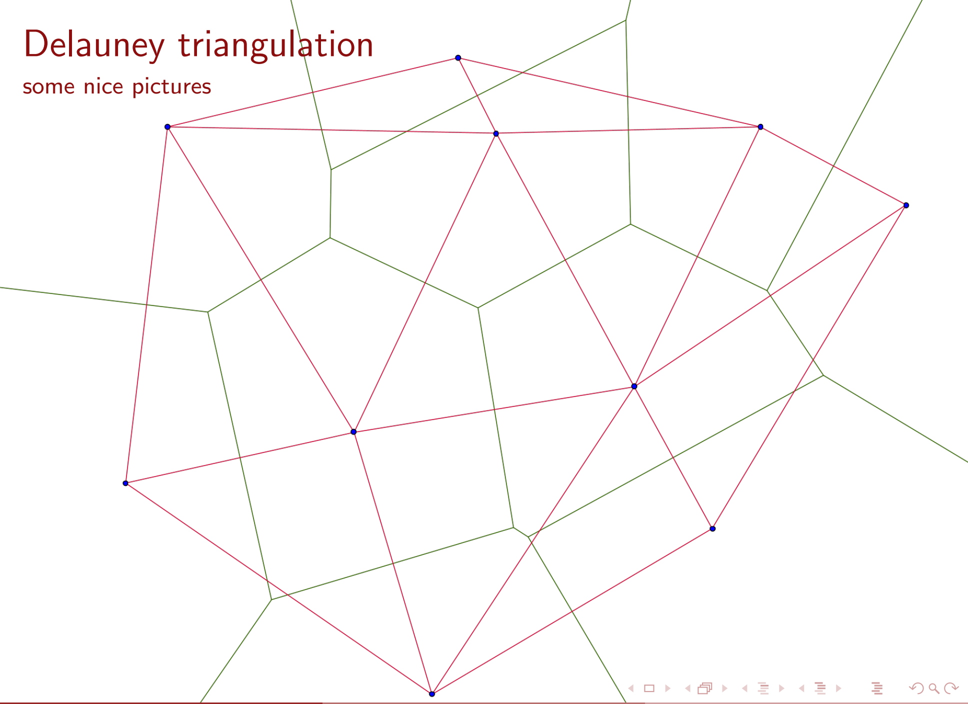
Voronoi diagram

some nice pictures



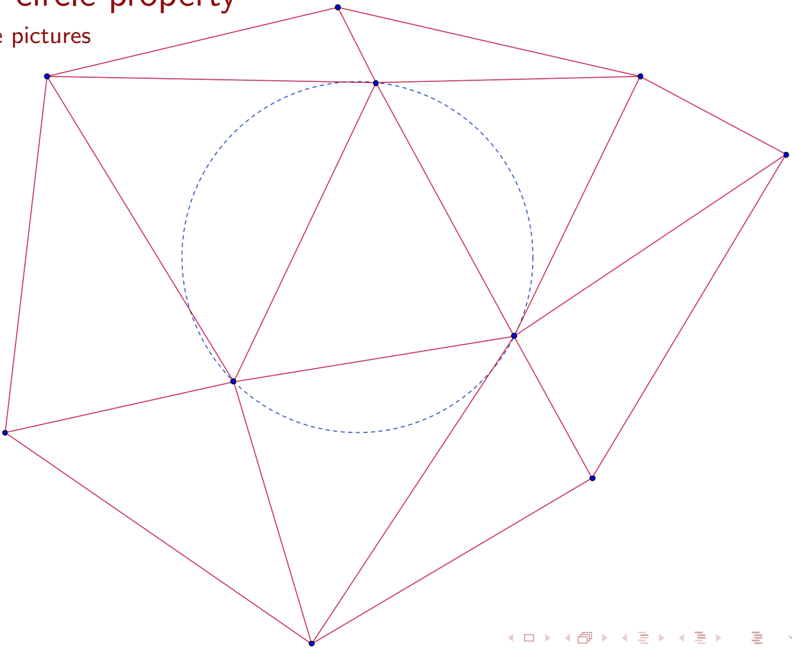
Delauney triangulation

some nice pictures



Empty circle property

some nice pictures



Reduction step

Consider $\Omega = \mathbb{R}^2 \setminus (Z \oplus B(0,\varepsilon))$, and $\mathcal{T}(Z)$ a Delauney triangulation and E a competitor for h, bounded.

$$\frac{p(E)}{|E|} = \frac{\sum_{\triangle \in \mathcal{T}_E(Z)} p(E,\triangle)}{\sum_{\triangle \in \mathcal{T}_E(Z)} |E \cap \triangle|} \geq \min_{\triangle \in \mathcal{T}_E(Z)} \frac{p(E,\triangle)}{|E \cap \triangle|} \geq \min_{\triangle \in \mathcal{T}_E(Z)} h_\varepsilon^*(\triangle)$$

and we have by the empty circle property

$$R(\triangle) \leq 1 + \varepsilon$$

The technical step

Theorem

Let $\triangle_{eq}$ be the equilateral triangle of circumradius $R(\triangle_{eq}) = 1$ then for $r < r^*$ small enough, for all triangle $\triangle$ satisfying $R(\triangle) = 1$, it holds

$$h_r^*(\triangle) \geq h_r^*(\triangle_{eq})$$

Idea of proof :

1. simple symmetry argument to rule out triangles with obtuse angles
2. prove the theorem "manually" for the others

Conclusion

The "real" theorem

Define the torsion rigidity $T(\Omega)$

$$T(\Omega) = \sup_{u \in H_0^1(\Omega)} \frac{\left(\int_\Omega u\right)^2}{\int_\Omega |\nabla u|^2}$$

Theorem (Buttazzo, Bucur, AdV - 2026)

For $\varepsilon < \varepsilon^*$ small enough, the hexagonal distribution maximizes the functional

$$E(\Omega) = \frac{T(\Omega)}{|\Omega|}$$

among sets $\Omega \in \mathcal{A}_\varepsilon$ of inradius $\rho(\Omega) = 1$.

Perspective : prove the same result for the eigenvalue $\lambda(\Omega)$.

Thank you for your attention