Shape optimization under inradius constraint

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What is a shape optimization problem ?

We consider the problem

$$\min \left\{ \mathcal{E}(\Omega) \;\middle|\; \Omega \in \mathcal{A} \right\}.$$

Here $\mathcal{A}$ is a class of subsets of $\mathbb{R}^d$ and is called the class of admissible shapes

Typically:

- Open subsets of $\mathbb{R}^d$
- Open subsets of $D \subset \mathbb{R}^d$
- Convex subsets of $\mathbb{R}^d$

Usual constraints:

- fixed volume
- fixed perimeter
- fixed inradius

Inradius constraint

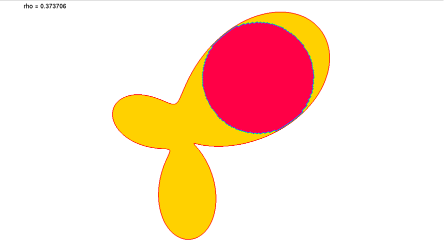
definition

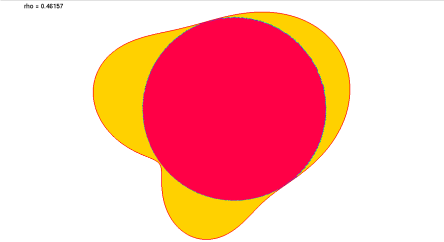
The problem we consider

$$\min\left\{\mathcal{E}(\Omega) \;\middle|\; \Omega \subset \mathbb{R}^2,\ \rho(\Omega) = 1\right\}.$$

$\Omega \subset \mathbb{R}^2$,

$$\rho(\Omega) = \sup\left\{r \;\middle|\; \exists x \in \Omega,\ B(x,r) \subset \Omega\right\}.$$





Optimization of the Cheeger constant with fixed inradius

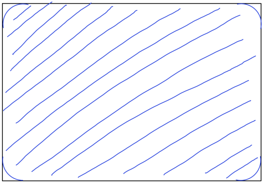
In collaboration with Dorin Bucur and Giuseppe Buttazzo

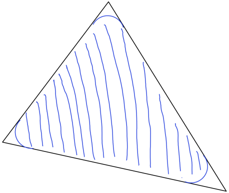
Cheeger constant

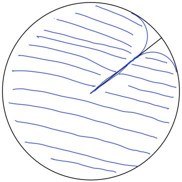
definition

$\Omega \subset \mathbb{R}^2$,

$$h(\Omega) = \inf \left\{ \frac{p(E)}{|E|} \;\middle|\; E \Subset \Omega \right\}$$







Cheeger constant

some properties

Basic properties :

- if $\Omega' \subset \Omega$, then $h(\Omega') \geq h(\Omega)$
- for all $t > 0$, $h(t\Omega) = t^{-1} h(\Omega)$

Link with the analytic problems :

- Alternative definition

$$h(\Omega) = \inf_{u \in W_0^{1,1}(\Omega)} \frac{\int_\Omega |\nabla u|}{\int_\Omega |u|}$$

- Faber Krahn type inequality

$$h(\Omega) \geq h(B_\Omega)$$

Optimization problem

not very interesting ...

Problems we look at :

1. $S = \sup h(\Omega)$ for $\rho(\Omega) = 1$
2. $I = \inf h(\Omega)$ for $\rho(\Omega) = 1$

Solutions are trivial :

1. $S = h(B^*)$
2. $I = 0$ with $\Omega^* = \mathbb{R}^2 \setminus \sqrt{2}\mathbb{Z}^2$

Optimization problem

modified setting

We introduce

$$\mathcal{A}_\varepsilon = \left\{ \Omega = \mathbb{R}^2 \setminus (Z + B(0,\varepsilon)) \;\middle|\; Z \subset \mathbb{R}^2,\ \text{discrete} \right\}$$

And consider

$$\inf \left\{ h(\Omega) \;\middle|\; \Omega \in \mathcal{A}_\varepsilon,\ \rho(\Omega) = 1 \right\}$$

Optimization problem

Pictural intuition

Optimization problem

much more interesting ...

Theorem (Bucur, Buttazzo, AdV)

Let $\Omega^* = \mathbb{R}^2 \setminus ((1+\varepsilon)Z_{hex} + B(0,\varepsilon))$, *then there exists* $\varepsilon^* > 0$ *such that for all* $0 < \varepsilon < \varepsilon^*$ *and* $\Omega \in \mathcal{A}_\varepsilon$ *satisfying* $\rho(\Omega) = 1$,

$$h(\Omega) \geq h(\Omega^*)$$

where Z_{hex} is the set of the centers of the hexagonal tiling of radius 1,

$$Z_{\text{hex}} = \left\{ \sqrt{3}n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{3}m \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \;\middle|\; (n,m) \in \mathbb{Z}^2 \right\}$$

Optimization problem

idea of the proof

1. Reduction step
   Use a Delauney triangulation and show
   $$h(\Omega) \geq \frac{1}{1+\varepsilon} \inf h^*_{r_\varepsilon}(\triangle)$$
   for $\triangle$ with circumradius $R(\triangle) = 1$
2. Solve the reduced problem
   For all $\triangle$ such that $R(\triangle) = 1$, and r small
   $$h^*_r(\triangle) \geq h^*_r(\triangle_{\text{eq}})$$

This strategy is similar to the strategy used by Bucur and Briani 2023 for solving the same problem with the infinite torsional rigidity.

Reduced Cheeger constant

who is h_r^* ?

For a triangle $\triangle$ of vertices A_i,

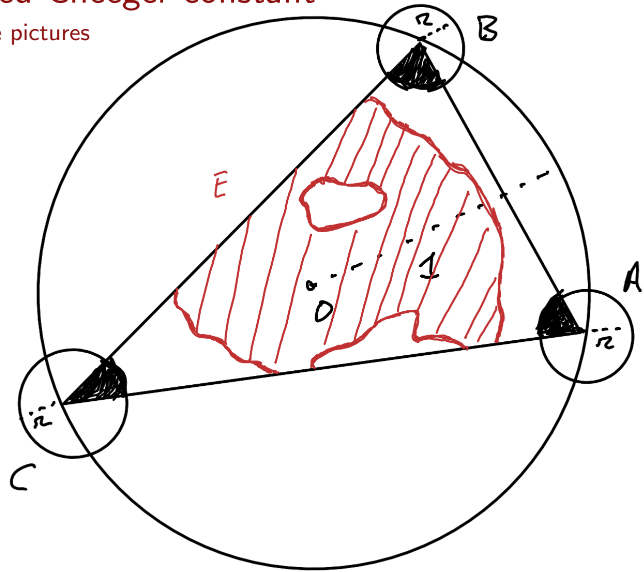
$$h_r^*(\triangle) = \inf \left\{ \frac{p(E,\triangle)}{|E|} \;\middle|\; E \subset \triangle \setminus \cup_i B(A_i, r) \right\}$$

and in analytic formulation

$$h_r^*(\triangle) = \inf \left\{ \frac{\int_\triangle |\nabla u|}{\int_\triangle |u|} \;\middle|\; u \in W^{1,1}(\triangle),\ u = 0 \text{ on } \cup_i B(A_i, r) \right\}$$

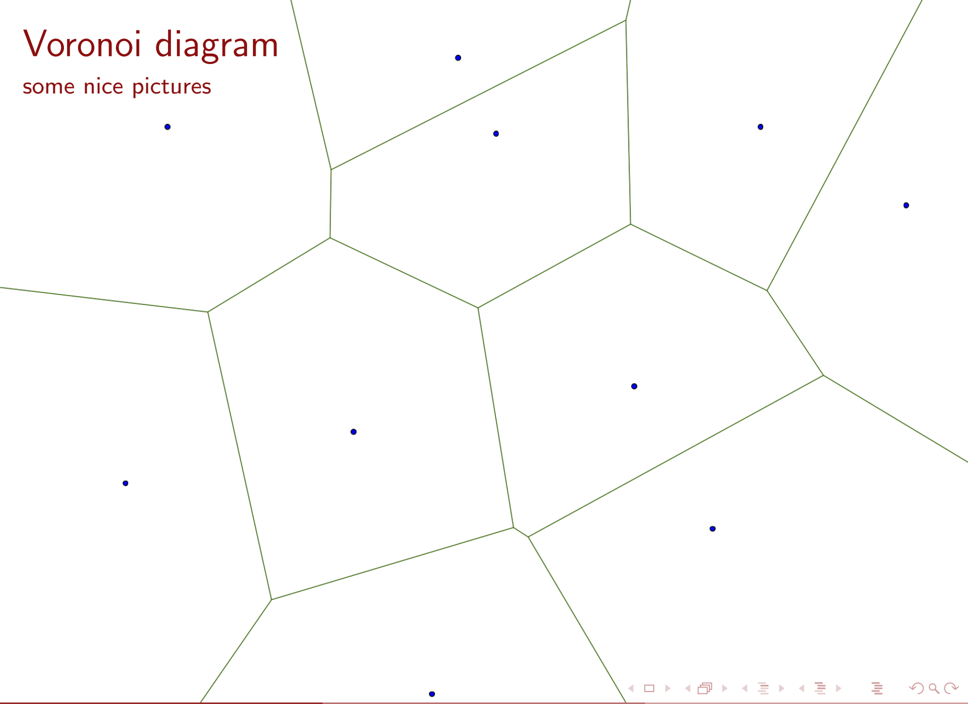
Reduced Cheeger constant

some nice pictures



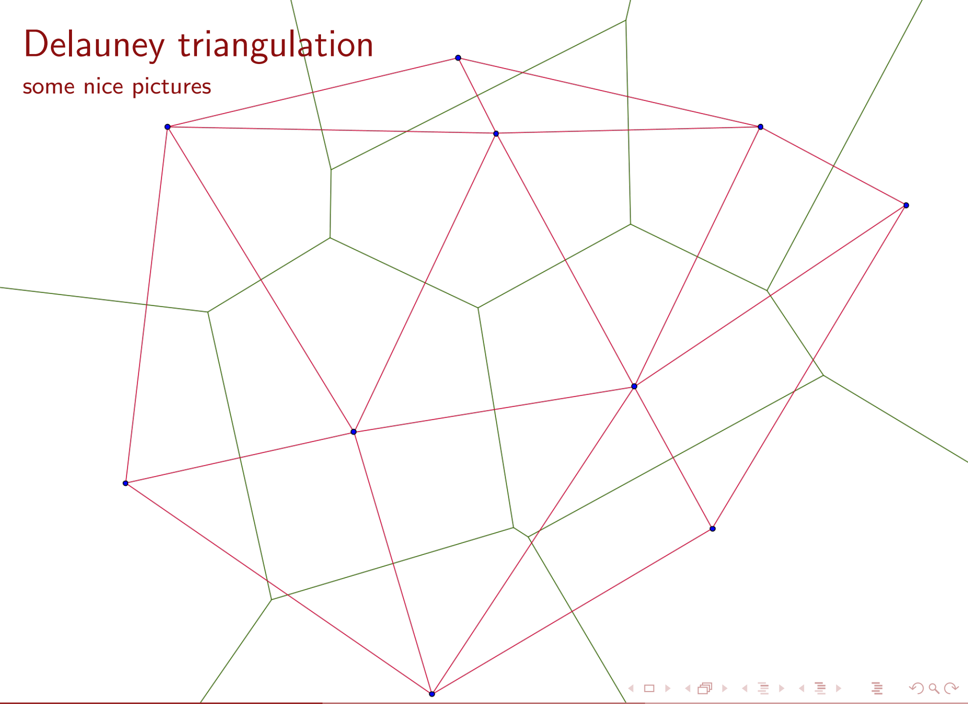
Voronoi diagram

some nice pictures



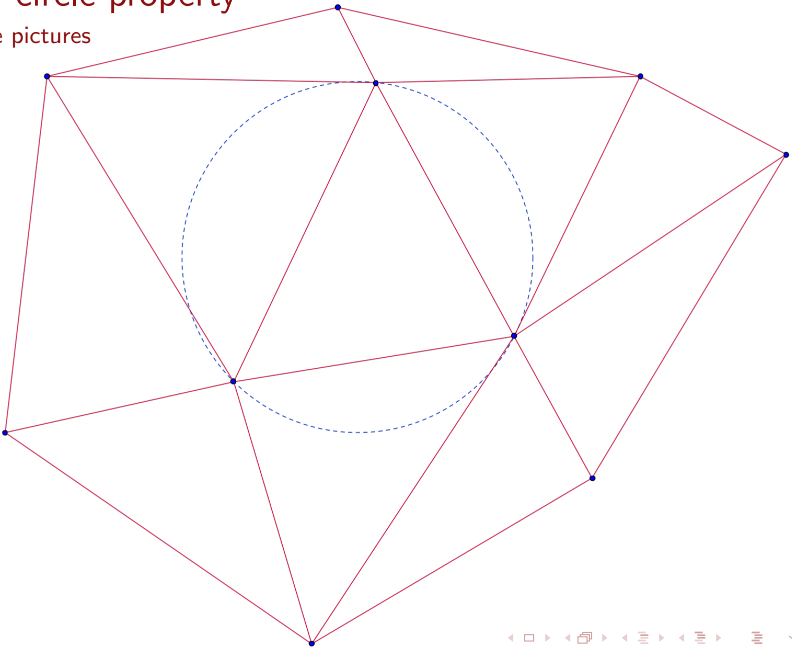
Delauney triangulation

some nice pictures



Empty circle property

some nice pictures



Reduction step

Consider $\Omega = \mathbb{R}^2 \setminus (Z \oplus B(0,\varepsilon))$, and $\mathcal{T}(Z)$ a Delauney triangulation and E a competitor for h, bounded.

$$\frac{p(E)}{|E|} = \frac{\sum_{\triangle \in \mathcal{T}_E(Z)} p(E,\triangle)}{\sum_{\triangle \in \mathcal{T}_E(Z)} |E \cap \triangle|} \geq \min_{\triangle \in \mathcal{T}_E(Z)} \frac{p(E,\triangle)}{|E \cap \triangle|} \geq \min_{\triangle \in \mathcal{T}_E(Z)} h_\varepsilon^*(\triangle)$$

and we have by the empty circle property

$$R(\triangle) \leq 1 + \varepsilon$$

The technical step

Theorem

Let $\triangle_{eq}$ be the equilateral triangle of circumradius $R(\triangle_{eq}) = 1$ then for $r < r^*$ small enough, for all triangle $\triangle$ satisfying $R(\triangle) = 1$, it holds

$$h_r^*(\triangle) \geq h_r^*(\triangle_{eq})$$

Idea of proof :

1. simple symmetry argument to rule out triangles with obtuse angles
2. prove the theorem "manually" for the others

The torsional rigidity case

the harder version

Define the torsional rigidity $T(\Omega)$

$$T(\Omega) = \int_\Omega w, \qquad w \in H^1_0(\Omega),\ -\Delta w = 1$$

Theorem (Buttazzo, Bucur, AdV - 2026)

For $\varepsilon < \varepsilon^*$ small enough, the hexagonal distribution maximizes the functional

$$E(\Omega) = \frac{T(\Omega)}{|\Omega|}$$

among sets $\Omega \in \mathcal{A}_\varepsilon$ of inradius $\rho(\Omega) = 1$.

Answers a conjecture of Buttazzo Santambrogio Varchon (2006)

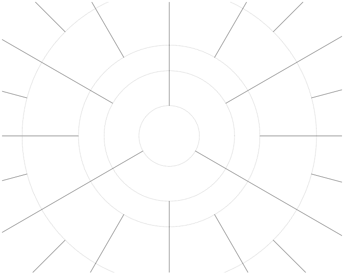
Numerical optimization for simply connected sets

In collaboration with Beniamin Bogosel and Dorin Bucur

The problem and the conjecture

$$\min \left\{ \mathcal{E}(\Omega) \;\middle|\; \Omega \subset \mathbb{R}^2,\ \text{open, simply connected},\ \rho(\Omega) = 1 \right\}$$

Conjecture (Bañuelos and Carroll): The optimal set is the Goodman domain



Levelset method

general ideas

1. Work in a square box.
2. Define the shape with a levelset function : $\Omega = \{\phi > 0\}$.
3. Compute the shape gradient v.
4. Compute the new levelset function ϕ^{new} by advection of ϕ in the direction of v.
$$\partial_t \varphi = v|\nabla \varphi|$$
5. Define the new shape : $\Omega^{\mathrm{new}} = \{\phi^{\mathrm{new}} > 0\}$.

Expected numerical difficulties

- Very fine strips $\to$ Hard to modelize
- Simple connectivity $\to$ Not very stable because of the strips
- Inradius constraint is local $\to$ Complicated to define a shape gradient

About the strip difficulties and simple connectivity

how to find them?

- Define the signed distance function of Ω,

$$\Psi_\Omega(x) = \begin{cases} \mathrm{dist}(x, \partial\Omega) & \text{if } x \in \Omega \\ -\mathrm{dist}(x, \partial\Omega) & \text{if } x \notin \Omega \end{cases}$$

- Define the skeleton of the shape

$$S_\Omega = \{x \mid \Psi_\Omega \text{ not differentiable in } x\}$$

- Look at $S_\Omega^\delta = S_\Omega \cap \{-\delta < \Psi_\Omega < 0\}$, where δ is the *carefully chosen* width

About the simple connectivity difficulties

how to force the constraint?

- Define at $\Omega_\delta = \{\Psi_\Omega > \delta\}$
- Transport the shape with the advection $\rightarrow \Omega_\delta^{\mathrm{new}}$
- compute the skeleton $S_{\Omega_\delta^{\mathrm{new}}}$
- Define $\Omega^{\mathrm{new}} = \{\Psi_{\Omega_\delta^{\mathrm{new}}} > -\delta\} \setminus S^\delta_{\Omega_\delta^{\mathrm{new}}}$.

Different from the Alexandrov Santosa log-barrier method.

About the inradius difficulties

how to differentiate it?

- Alternative definition :

$$\rho(\Omega) = \sup_{x \in \Omega} \Psi_\Omega$$

- Approximation, with L^p norms

$$\rho(\Omega) = \left(\int_\Omega (\Psi_\Omega(x))^p \right)^{1/p}$$

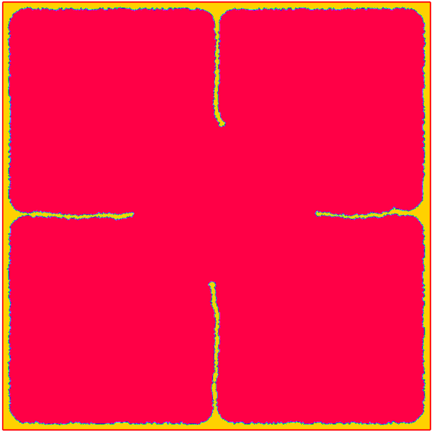
- Expression of the shape gradient G : (Feppon, Allaire, Dapogny 2020)

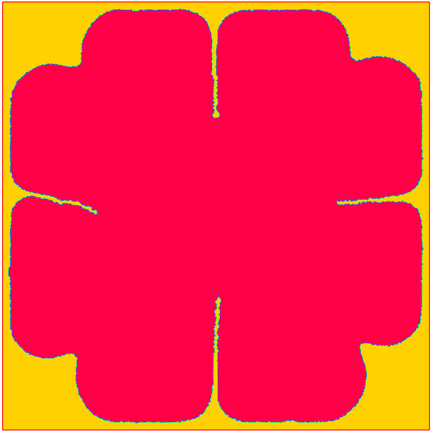
$$\int_{\partial\Omega} Gu + \int_\Omega \omega(\nabla G \cdot \nabla \Psi_\Omega)(\nabla u \cdot \nabla \Psi_\Omega) = -\int_\Omega p \Psi_\Omega^{p-1} u$$

The general scheme

1. compute $\Omega = \{\phi > 0\}$
2. compute the gradients using the approximation for the inradius V
3. regularize the gradient, $V \to V^{\mathrm{reg}}$
4. Advect $\Psi_\Omega - \delta$ along V^{reg}, obtain $\phi_\delta^{\mathrm{new}}$
5. compute $\Omega_\delta^{\mathrm{new}} = \{\phi_\delta^{\mathrm{new}} > 0\}$
6. compute $\Psi_{\Omega_\delta^{\mathrm{new}}}$ and $S^\delta_{\Omega_\delta^{\mathrm{new}}}$
7. compute $\Omega^{\mathrm{new}} = \{\Psi_{\Omega_\delta^{\mathrm{new}}} > -\delta\} \setminus S^\delta_{\Omega_\delta^{\mathrm{new}}}$
8. set $\phi^{\mathrm{new}} = \Psi_{\Omega^{\mathrm{new}}}$

Some early results





Thank you for your attention