

Optimality of the hexagonal tiling for the torsional rigidity and the first eigenvalue with fixed inradius in 2D

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Main problems

We work in \mathbb{R}^2 , and we are interested in the study of the two problems for $2 < p < +\infty$:

$$\min \{ \lambda_p(\Omega) \mid \Omega \subset \mathbb{R}^2, \text{ open}, \rho(\Omega) = 1 \},$$
$$\sup \left\{ \frac{T_p(\Omega)}{|\Omega|} \mid \Omega \subset \mathbb{R}^2, \text{ open}, |\Omega| < +\infty, \rho(\Omega) = 1 \right\}.$$

where,

- $\rho(\Omega)$ is the inradius of Ω ,
- $\lambda_p(\Omega)$ is the principal eigenvalue of the p -Laplacian of Ω with Dirichlet boundary conditions,
- $T_p(\Omega)$ is the p -torsional rigidity of Ω .

- Principal eigenvalue of the p -Laplacian of Ω with Dirichlet boundary conditions :

$$\lambda_p(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} |u|^p} \mid u \in W_0^{1,p}(\Omega) \right\} = \int_{\Omega} |\nabla u_p|^p,$$

where u_p is the associated eigenfunction and is the unique solution to

$$\begin{cases} -\Delta_p u_p = \lambda_p(\Omega) u_p^{p-1} \text{ in } \Omega, \\ u_p \in W_0^{1,p}(\Omega), \int_{\Omega} |u|^p = 1. \end{cases}$$

- p -torsional rigidity of Ω ,

$$T_p(\Omega) = \sup \left\{ \frac{(\int_{\Omega} |u|)^{\frac{p}{p-1}}}{(\int_{\Omega} |\nabla u|^p)^{\frac{1}{p-1}}} \mid u \in W_0^{1,p}(\Omega) \right\} = \int_{\Omega} w_p,$$

where w_p is the torsion function and is the unique solution to

$$\begin{cases} -\Delta_p w_p = 1 \text{ in } \Omega, \\ w_p \in W_0^{1,p}(\Omega). \end{cases}$$

Known results

- For the eigenvalue problem we have an existence result:

Theorem (Bucur, Buttazzo, V - 2025)

For any p such that $2 < p < +\infty$, there exists and Ω_{opt} of inradius 1, minimizer for the eigenvalue. And moreover, $\Omega_{opt} = \mathbb{R}^2 \setminus Z$ where Z is a discrete set with no accumulation points

- For the torsion problem, a limit result has been proven by Briani and Bucur,

Theorem (Briani, Bucur - 2023)

For all Ω open subset of \mathbb{R}^2 of finite measure, and inradius 1

$$\frac{T_\infty(\Omega)}{|\Omega|} = \frac{\int_\Omega d(x, \partial\Omega)}{|\Omega|} \leq \lim_{N \rightarrow +\infty} \frac{T_\infty(B(0, N) \cap (\mathbb{R}^2 \setminus Z_{hex}))}{\pi N^2}$$

where Z_{hex} is the set of the centers of the hexagonal tiling of inradius 1.

Question : Can we prove for all $2 < p < +\infty$, that for any Ω of inradius 1 and finite measure,

- $\lambda_p(\Omega) \leq \lambda_p(\mathbb{R}^2 \setminus Z_{hex})$,
- $\frac{T_p(\Omega)}{|\Omega|} \leq \lim_{N \rightarrow +\infty} \frac{T_\infty(B(0,N) \cap (\mathbb{R}^2 \setminus Z_{hex}))}{\pi N^2}$?

Answer : Not yet

Reduction to a problem on triangles

We can show that the two problems can be reduced to a problem on triangles.

We define for any triangle Δ of vertices A_1, A_2, A_3 ,

- $\lambda_p^*(\Delta) = \inf \left\{ \frac{\int_{\Delta} |\nabla u|^p}{\int_{\Delta} |u|^p} \mid u \in W^{1,p}(\Delta), \forall i, u(A_i) = 0 \right\},$
- $T_p^*(\Delta) = \sup \left\{ \frac{(\int_{\Delta} |u|)^{\frac{p}{p-1}}}{(\int_{\Delta} |\nabla u|^p)^{\frac{1}{p-1}}} \mid u \in W^{1,p}(\Delta), \forall i, u(A_i) = 0 \right\}.$

and we show for any Ω open of inradius 1

- $\lambda_p(\Omega) \geq \inf \{ \lambda_p^*(\Delta) \mid R(\Delta) = 1 \}$
- $\frac{T_p(\Omega)}{|\Omega|} \leq \sup \left\{ \frac{T_p^*(\Delta)}{|\Delta|} \mid R(\Delta) = 1 \right\}$

with equality if the optimizers are the equilateral triangle and $\Omega = \mathbb{R}^2 \setminus Z_{hex}$.

Proof of the reduction step for the Eigenvalue problem

(not rigorous)

Let $\Omega = \mathbb{R}^2 \setminus Z$ be a competitor for λ_p , with Z discrete with no accumulation points. Define \mathcal{F} a Delaunay triangulation of Z , then

$$\begin{aligned}\lambda_p(\Omega) &= \frac{\sum_{\Delta \in \mathcal{F}} \int_{\Delta} |\nabla u_p|^p}{\sum_{\Delta \in \mathcal{F}} \int_{\Delta} |u_p|^p} \\ &\geq \inf_{\Delta \in \mathcal{F}} \frac{\int_{\Delta} |\nabla u_p|^p}{\int_{\Delta} |u_p|^p} \\ &\geq \inf_{\Delta \in \mathcal{F}} \lambda_p^*(\Delta) \\ &\geq \inf \{ \lambda_p^*(\Delta) \mid R(\Delta) = 1 \}\end{aligned}$$

Natural equivalents of the problems for $p = 2$

Problem : When $p = 2$, the 2-capacity of points is null and the eigenvalue and the torsional rigidity don't see them.

A solution to this problem is to work, for $\varepsilon > 0$, in the class

$$\mathcal{A}_\varepsilon = \left\{ \Omega = \mathbb{R}^2 \setminus \bigcup \overline{B(x_i, \varepsilon)} \mid (x_i) \in \mathbb{R}^{\mathbb{N}}, \text{ with no accumulation points} \right\},$$

and we consider the problems

- $\inf \{ \lambda_2(\Omega) \mid \Omega \in \mathcal{A}_\varepsilon, \rho(\Omega) = 1 \},$
- $\sup \left\{ \frac{T_2(\Omega)}{|\Omega|} \mid \Omega \in \mathcal{A}_\varepsilon, |\Omega| < +\infty, \rho(\Omega) = 1 \right\}.$

Remark

This setting is not new, it was already considered for problems arising in economics by Morgan and Bolton in 2002, or in 2006 in paper from Buttazzo, Santambrogio and Varchon. And in both cases, the conjecture was that the hexagonal tiling should be the optimal distribution.

Equivalents of the problem on triangles for $p = 2$

With the same procedure, we show that we can reduce the problems to two problems on triangles

- $\inf \{ \lambda_r^*(\Delta) \mid R(\Delta) = 1 \}$
- $\sup \left\{ \frac{T_r^*(\Delta)}{|\Delta|} \mid R(\Delta) = 1 \right\}$

with, for any triangle Δ of vertices A_1, A_2, A_3 and any $r > 0$,

$$\lambda_r^*(\Delta) = \inf \left\{ \frac{\int_{\Delta} |\nabla u|^2}{\int_{\Delta} |u|^2} \mid u \in H^1(\Delta), \forall i, u = 0 \text{ on } (\partial B(A_i, \varepsilon)) \cap \Delta \right\},$$

$$T_r^*(\Delta) = \sup \left\{ \frac{\left(\int_{\Delta} |u| \right)}{\int_{\Delta} |\nabla u|^2} \mid u \in H^1(\Delta), \forall i, u = 0 \text{ on } (\partial B(A_i, \varepsilon)) \cap \Delta \right\}.$$

A first theorem

We show the following for the torsion problem

Theorem

For any acute or right triangle Δ , of circumradius $R(T) = 1$, it holds

$$\lim_{r \rightarrow 0} \frac{1}{|\ln(r)|} \frac{T_r^*(\Delta)}{|\Delta| - \pi/2 r^2} = \frac{|\Delta|}{\pi}.$$

And as a corollary, **with an additional symmetry argument to rule out obtuse triangles**, we obtain that in the limit the quantity,

$$\frac{1}{|\ln(r)|} \frac{T_r^*(\Delta)}{|\Delta| - \pi/2 r^2},$$

is maximal for the equilateral triangle.

A second theorem

About the eigenvalue problem, we show

Theorem

For any triangle Δ , it holds

$$\lim_{r \rightarrow 0} \lambda_r^*(\Delta) T_r^*(\Delta) = |\Delta|.$$

And therefore, we obtain that the quantity,

$$\ln(r) \lambda_r^*(\Delta),$$

is minimal in the limit for the equilateral triangle

$p = 1$, the Cheeger case, the setting

For any open set Ω in \mathbb{R}^2 , we define the Cheeger constant of Ω , as

$$h(\Omega) = \inf \left\{ \frac{\rho(E)}{|E|} \mid E \Subset \Omega \right\}.$$

It is possible to show that actually

$$h(\Omega) = \lambda_1(\Omega).$$

Therefore in the same spirit as before we want to solve for $\varepsilon > 0$,

$$\min \{ h(\Omega) \mid \Omega \in A_\varepsilon, \rho(\Omega) = 1, \}$$

and once again, this problem can be reduced to a problem on triangles, which writes for $r > 0$,

$$\min \{ h_r^*(\Delta) \mid R(\Delta) = 1 \},$$

with, for any triangle Δ of vertices A_1, A_2, A_3

$$h_r^*(\Delta) = \inf \left\{ \frac{\rho(E, \Delta)}{|E|} \mid E \subset \Delta \setminus \bigcup_i B(A_i, r) \right\}.$$

$p = 1$, the Cheeger case, the results

Theorem

There exists an $r^ > 0$ (explicit) such that for any $r < r^*$, it holds for any triangle Δ of circumradius $R(\Delta) = 1$,*

$$h_r^*(\Delta) \geq h_r^*(\Delta_{eq}),$$

where Δ_{eq} is the equilateral triangle of circumradius 1.

As a corollary, we obtain the result in the whole space

Theorem

There exists ε^ sufficiently small (explicit) such that for any $\varepsilon < \varepsilon^*$, it holds for any set $\Omega \in \mathcal{A}_\varepsilon$, of inradius $\rho(\Omega) = 1$,*

$$h(\Omega) \geq h \left(\mathbb{R}^2 \setminus \bigcup_{x \in (1+\varepsilon)Z_{hex}} B(x, \varepsilon) \right)$$

Thank you very much for your attention !!