Optimality for a generalised Cheeger inequality in 2D

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Setting of the problem

We are interested in studying for $1 \leq q < p \leq +\infty$ the functionnal,

$$\mathcal{F}_{p,q}(\Omega) = \frac{\lambda_p(\Omega)^{1/p}}{\lambda_q(\Omega)^{1/q}},$$

defined for any $\Omega \in \mathbb{R}^d$.
We are interested in the problems

$$\min\left\{\mathcal{F}_{p,q}(\Omega) \;\middle|\; \Omega \subset \mathbb{R}^d\right\} \qquad \max\{\mathcal{F}_{p,q}(\Omega) \;\middle|\; \Omega \subset \mathbb{R}^d\}$$

Where $\lambda_p(\Omega)$ represents the principal eignevalue of the p-Laplacien with Dirichlet boundary conditions on Ω, it can be defined as follows

$$\lambda_p(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

General properties of the principal eigenvalue of the p-Laplacian

- $\forall \Omega \subset \mathbb{R}^d$, satisfying $|\Omega| < +\infty$, $\lambda_p(\Omega)|\Omega|^{p/d} \geq \lambda_p(B)|B|^{p/d}$
- $\forall \Omega \subset \mathbb{R}^d$, $\forall t > 0$, $\quad \lambda_p(t\Omega) = t^{-p}\lambda_p(\Omega)$
- $\forall \Omega \subset \mathbb{R}^d$, $\quad \lim_{p \to +\infty} \lambda_p(\Omega)^{1/p} = \rho(\Omega)^{-1}$
- $\lambda_1(\Omega) = h(\Omega)$ where $h(\Omega)$ is the cheeger constant of Ω,

$$h(\Omega) = \inf \left\{ \frac{p(E)}{|E|} \;\middle|\; E \Subset \Omega \right\}.$$

Existence results

Theorem

There exist two sets Ω_m and Ω_M respectively minimizing and maximizing $\mathcal{F}_{p,q}$ and the supremum is finite if and only if $q > d$.

Theorem

If $d < q < p \leq +\infty$, there exist two discrete sets X_m and X_M such that their respective complementary $\Omega_m = \mathbb{R}^d \setminus X_m$ and $\Omega_M = \mathbb{R}^d \setminus X_M$ are respectively minimizer and maximizer of $\mathcal{F}_{p,q}$

The problem in 2D

Question : Can we show that the maximum is attained when X_M is the set of the centers of a regular hexagonal tiling ?

- For general p and q maybe the case but clearly too hard at the moment.
- For the limit case $p = +\infty$ and $2 < q < +\infty$, we have numerical evidence and we hope to be able to show it.
- At the moment we have a proof for $p = +\infty$ and $q = 1$ for a similar problem where we remove small disks instead of points

Reduction to a problem on triangles

In the general case, $p = +\infty$, $2 < q < +\infty$, the problem

$$\max\left\{\mathcal{F}_{+\infty,q}(\Omega) \;\middle|\; \Omega \subset \mathbb{R}^d\right\}$$

is equivalent to the problem

$$\min\left\{\lambda_q(\Omega) \;\middle|\; \Omega \subset \mathbb{R}^d,\ \rho(\Omega) = 1\right\} = \min\left\{\lambda_q(\Omega) \;\middle|\; \Omega = \mathbb{R}^d \backslash X,\ \rho(\Omega) = 1\right\}.$$

Let $\mathcal{A}$ be a Delauney triangulation of X, we have

$$\lambda_q(\mathbb{R}^d \setminus X) \geq \inf_{T \in \mathcal{A}} \lambda_q^*(T)$$

with

$$\lambda_q^*(T_{A,B,C}) = \inf\left\{\frac{\int_T |\nabla u|^q}{\int_T |u|^q} \;\middle|\; u \in W^{1,q}(T),\ u(A) = u(B) = u(C) = 0\right\}$$

General problem on triangles

Question : Is the equilateral triangle a solution to the following problem:

$$\min \left\{ \lambda_q^*(T) \;\middle|\; R(T) = 1 \right\}.$$

- For general $2 < q < +\infty$ we have numerical evidence but the proof is not complete.
- $q = +\infty$ it is the case for a related problem involving the torsional rigidity.
- $q = 1$ it is the case for the Cheeger constant by removing small disks instead of points.
- $q = 2$ it is the case in the limit problem when the radius of the small balls goes to zero for the torsional rigidity problem.

We want to solve a similar problem but in the easier case of the Cheeger constant, we consider for some $r > 0$,

$$\min \left\{ h(\Omega) \;\middle|\; \Omega = \mathbb{R}^d \setminus \bigcup_i B(x_i, r),\ \rho(\Omega) = 1 \right\}$$

It can be reduced to studying

$$min\left\{ h_r^*(T) \;\middle|\; R(T) = 1 \right\}$$

with

$$h_r^*(T_{A,B,C}) = \left\{ \frac{p(E,T)}{|E|} \;\middle|\; E \subset T,\ E \supset (B(A,r) \cup B(B,r) \cup B(C,r)) \cap T \right\}$$

Theorem

If $r < r^*$ (explicit), the equilateral triangle minimizes h_r^* among all triangles of circumradius 1.

proof - On the board $\Longrightarrow$

Corollary

For any $\varepsilon < \varepsilon^*$ (explicit), the set $\Omega_{hex,\varepsilon} = \mathbb{R}^d \setminus \bigcup_{x \in X_{hex,\epsilon}} B(x, \varepsilon)$ is minimal for the Cheeger constant among all admissible sets of inradius 1.