Optimality of the honeycomb structure in some shape optimization problems

Alexis de Villeroché

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What is a shape optimization problem ?

We consider the problem

$$\min \left\{ F(\Omega) \middle| \Omega \in \mathcal{A} \right\}. \tag{1}$$

Here $\mathcal{A}$ is a class of subsets of $\mathbb{R}^d$ and is called the class of admissible shapes

Typically:

- Open subsets of $\mathbb{R}^d$
- Open subsets of $D \subset \mathbb{R}^d$
- Convex subsets of $\mathbb{R}^d$

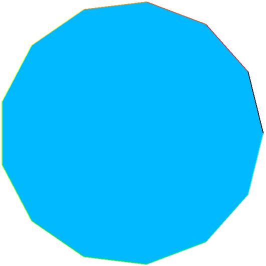
Usual constraints:

- fixed volume
- fixed perimeter
- fixed inradius

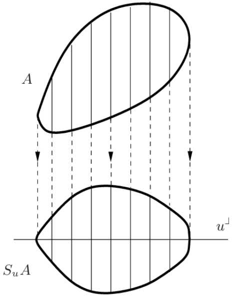
If $|\Omega| = |B|$,

$$P(\Omega) \geq P(B)$$

Regular polygon : $P_n = 2\sqrt{n \frac{\sin(\pi/n)}{\cos(\pi/n)}}$



Steiner symmetrization

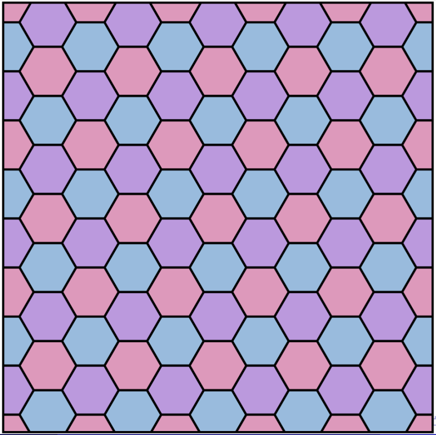


The historic problems

The honeycomb theorem

Theorem (Thomas C. Hales - 1999)

Among all partitions of the plane in cells of equal area the hexagonal tesselation (or honeycomb structure) has the least average perimeter



Minimization of the first Dirichlet eigenvalue

The lowest frequency of a drum of given area

- first eigenvalue of the Dirichlet Laplacian

$$\lambda_1(\Omega) = \inf_{v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2}$$

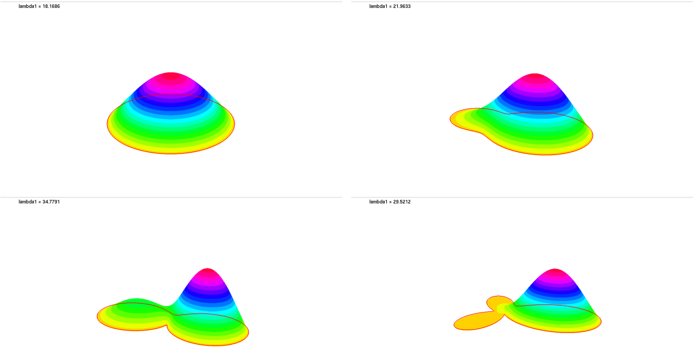
- Associated eigenfunction

$$\begin{cases} -\Delta u & = \lambda_1(\Omega)\, u \quad \text{in } \Omega \\ u & = 0 \quad \text{on } \partial\Omega \\ \int_\Omega u^2 & = 1 \end{cases}$$

- Faber-Krahn inequality (1920s), if $|\Omega| = |B|$:

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

Pictures of eigenfunctions



Maximization of the Torsionnal rigidity of Ω

The resistance to torsion of a bar of cross section Ω

- Torsion function of Ω

$$\begin{cases} -\Delta w &= 1 \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{cases}$$

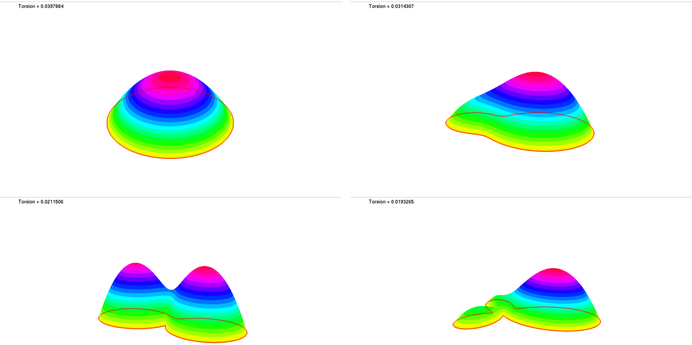
- Torsionnal rigidity of Ω

$$T(\Omega) = \int_\Omega w = \max_{v \in H_0^1(\Omega)} \int_\Omega 2\,v - \int_\Omega |\nabla v|^2$$

- Saint Venant inequality (1951), if $|\Omega| = |B|$:

$$T(\Omega) \leq T(B)$$

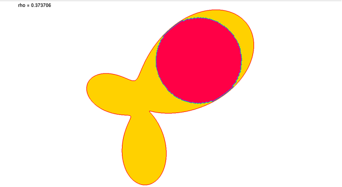
Pictures of torsion functions

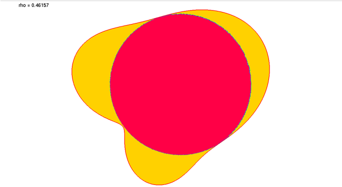


Inradius

Let Ω be an open set, define $\rho(\Omega)$ its inradius

$$\rho(\Omega) = \inf\left\{r \;\middle|\; \exists x \in \Omega,\ B(x,r) \subset \Omega\right\}.$$





Torsion problem with fixed inradius constraint

We work in the class

$$\mathcal{A}_\varepsilon = \left\{ \Omega = \mathbb{R}^2 \setminus \bigcup B(x_i, \varepsilon) \;\middle|\; (x_i) \in \mathbb{R}^{2^{\mathbb{N}}} \text{ with no accumulation points} \right\}$$

and we consider

$$\sup \left\{ \frac{T(\Omega)}{|\Omega|} \;\middle|\; \Omega \in \mathcal{A}_\varepsilon,\ \rho(\Omega) = 1,\ |\Omega| < +\infty \right\}$$

Question : do we obtain the honeycomb structure as a maximizer ?

A few pictures about what we actually compute

Reduction to a problem on triangles

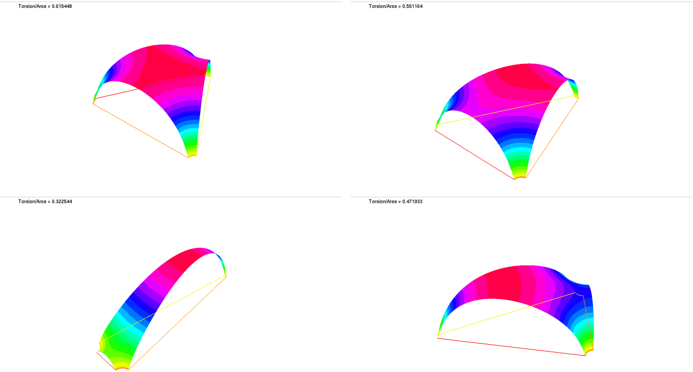
We show that we can reduce the problem in the full plane to a problem on triangles.
We define for a triangle $\triangle$ of vertices A_1, A_2, A_3, the modfied torsion function as the solution to

$$\begin{cases} -\Delta w_r^* & = 1 \quad \text{in } \triangle \\ w_r^* & = 0 \quad \text{on } \partial B(A_i, r) \cap \triangle \\ \partial_n w_r^* & = 0 \quad \text{on } \partial \triangle \setminus \bigcup B(A_i, r) \end{cases}$$

and the modified torsionnal rigidity

$$T_r^*(\triangle) = \int_\triangle w_r^*$$

A few pictures about the triangle problem



The honeycomb structure is optimal in the limit

Theorem (Bucur, Buttazzo, V. - 2025)

For any acute or right triangle $\triangle$, we have the following limit

$$\lim_{t \to 0} \frac{T_r^*(\triangle)}{|\ln(r)|} = \frac{|\triangle|^2}{\pi}$$

As a corollary we obtain, **with an additionnal symmetry argument to rule out obtuse triangles**, that the equilateral triangle is optimal in the limit.

Remark - We obtain the same result for the eigenvalue.

Thank you very much for your attention !!

- circular drum video: https://www.youtube.com/watch?v=QksHbCwYngw
- string modes video : https://www.youtube.com/watch?v=cnH2ltfW48U