Shape Optimization
Phd Seminar

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What is a shape optimization problem ?
(That is the question)

We consider the problem

$$\min \left\{ F(\Omega) \middle| \Omega \in \mathcal{A} \right\}. \tag{1}$$

Here $\mathcal{A}$ is a class of subsets of $\mathbf{R}^d$ and is called the class of admissible shapes

Typically:

- Open subsets of $\mathbf{R}^d$
- Open subsets of $D \subset \mathbf{R}^d$
- Convex subsets of $\mathbf{R}^d$

Usual constraints:

- fixed volume
- fixed perimeter

Eigenvalue and eigenfunction of the Dirichlet Laplacian

(First slide with definitions)

- Eigenvalue of the Dirichlet Laplacian

$$\lambda_k(\Omega) = \inf_{V_k \subset H_0^1(\Omega)} \sup_{v \in V_k \subset \{0\}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2}$$

with V_k a subspace of dimension k of $H_0^1(\Omega)$

- Associated eigenfunction

$$\begin{cases} -\Delta u_k & = \lambda_k(\Omega)\, u_k & \text{in } \Omega \\ u_k & = 0 & \text{on } \partial\Omega \\ \int_\Omega {u_k}^2 & = 1 & \end{cases}$$

Torsion and torsion function of Ω

(Second slide with definitions)

- Torsion function of Ω

$$\begin{cases} -\Delta w & = 1 \quad \text{in } \Omega \\ w & = 0 \quad \text{on } \partial\Omega \end{cases}$$

- Torsion of Ω

$$T(\Omega) = \int_\Omega w = \max_{v \in H_0^1(\Omega)} \int_\Omega 2\,v - \int_\Omega |\nabla v|^2$$

Classical Examples

(The solution is always the ball)

- Faber-Krahn inequality

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

- Krahn-Szego inequality

$$\lambda_2(\Omega) \geq \lambda_2(\Theta)$$

- Saint Venant theorem

$$T(\Omega) \leq T(B)$$

- Kohler Jobin inequality

$$\lambda_1(\Omega)^{\frac{d+2}{2}}\, T(\Omega) \geq \lambda_1(B)^{\frac{d+2}{2}}\, T(B)$$

Existence of solutions - Direct method in calculus of variation

(When everything goes well)

Consider the problem

$$\min_{u \in H} J(u)$$

- Take (u_n) a minimizing sequence $(J(u_n) \to infJ)$
- Prove that (u_n) converges in some sense to u in H
- Prove that J is lower semi continuous for this convergence

Then u is a minimizer of J in H.

γ-convergence
(Third slide with definitions)

We say that a sequence of sets (Ω_n) γ-converges to Ω if the associated torsion functions sequence (w_{Ω_n}) converges strongly in L^2 to the torsion function w_Ω of Ω

$$\Omega_n \xrightarrow{\gamma} \Omega \text{ iff } \int_{\mathbf{R}^d} |w_{\Omega_n} - w_\Omega|^2 \to 0$$

Problems with the γ-convergence and relaxation

(When not everything goes well)

Take a sequence of open sets (Ω_n) can we say when it converges ? and to what ?

Notice that if (w_n) is bounded in $H^1(\mathbf{R}^d)$ then it has a limit w in L^2

Define the measure

$$\mu = \frac{1 + \Delta w}{w}$$

Then w solves

$$\begin{cases} -\Delta w + w\,\mu = 1 & \text{in } [H^1(\mathbf{R}^d) \cap L^2(\mu)]' \\ w \in H^1(\mathbf{R}^d) \cap L^2(\mu) & \end{cases}$$

Capacity and capacitary measures
(Fourth slide with definitions)

- Capcity of Ω

$$Cap(\Omega) = \inf \left\{ \int (|\nabla u|^2 + u^2) \middle| u \in U_\Omega \right\}$$

with $U_\Omega = \left\{ u \in H^1(\mathbf{R}^d) \text{ and } u \geq 1 \text{ in a neighborhood of } \Omega \right\}$

- Ω is said to be quasi-open if it is open up to a zero capacity set
- μ is said to be a capacitary measure if it doesn't charge sets of zero capacity
  example :

$$\infty_\Omega(\tilde{\Omega}) = \begin{cases} 0 & \text{if } \tilde{\Omega} \subset \Omega \text{ (up to a zero capacity set)} \\ +\infty & \text{otherwise} \end{cases}$$

Buttazzo-Dal Maso Existence theorem
(Finally !!)

Theorem

Let $\mathcal{A} = \{\Omega \mid \Omega$ *quasi open set,* $\Omega \subset D\}$ *with* D *open and bounded, and let* $F : \mathcal{A} \to \mathbf{R}$ *be a* γ*-lower semi continuous functional, decreasing for the inclusion, then the problem*

$$\min \left\{ F(\Omega) \middle| \Omega \in \mathcal{A},\ |\Omega| = c \right\}$$

admits at least a solution for $0 < c \leq |D|$*.*

PL Lions Concentration-Compactness principle for shapes

(Not everything always goes where we want but we can fin it)

Theorem

Let (Ω_n) *be a sequence of quasi open sets of* $\mathbf{R}^d$ *with uniformly bounded measure, there exists a subsequence satisfying one of the two following points*

- *(Compactness) there exits a sequence* (y_n) *of elements of* $\mathbf{R}^d$ *and a capacitary measure* μ *such that* $y_n + \Omega_n \xrightarrow{\gamma} \mu$*,*
- *(Dichotomy) There exists a sequence of subsets* $(\tilde{\Omega}_n)$ *such that*

$$\|w_{\Omega_n} - w_{\tilde{\Omega}_n}\|_{L^2} \to 0 \quad \text{and} \quad \tilde{\Omega}_n = \tilde{\Omega}_{n,1} \cup \tilde{\Omega}_{n,2} \subset \Omega_n$$

with $dist(\tilde{\Omega}_{n,1}, \tilde{\Omega}_{n,2}) \to +\infty$ *and* $\liminf |\tilde{\Omega}_{n,i}| > 0$*.*

Back to the eigenvalues - some recent results

(It's not always the ball)

For every k, λ_k admits a minimizer Ω_k^*

- Ω_k^* is bounded
- Ω_k^* has finite perimeter
- For all $k \geq 5$, Ω_k^* is not a ball or a union of balls

BUT we don't know the minimizer if $k \geq 2$ and we don't have a proof yet that for all k, Ω_k^* is an open set.

Stability of the full spectrum

(The work I'm doing at the moment)

Question : If Ω is such that $\lambda_l(\Omega)$ is close to $\lambda_l(\Omega_l^*)$ what can we say about the other eigenvalues ?

We only have answers for λ_1 and λ_2

- for λ_1 we have the sharp result (Bucur Lamboley Nahon Prunier 2023)

$$|\lambda_k(\Omega) - \lambda_k(B)| \leq C\,\lambda_1(\Omega)^{\frac{1}{2}}(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{2}}$$

- for λ_2 I have proved the non sharp result

$$\begin{aligned}|\lambda_k(\Omega) - \lambda_k(\Theta)| \leq \quad & C\,k^{2+\frac{4}{d}}\,|\Omega|^{1+\frac{2}{d}}\,\lambda_2(\Omega)^{2+\frac{d}{2}} \\ & \Big[|\Omega|^{1+\frac{2}{d}}\,\lambda_2(\Omega)^{\frac{d}{2}}\,(\lambda_2(\Omega) - \lambda_2(\Theta)) \\ & + |\Omega|^{\frac{2}{d\,(d+1)}}\,(\lambda_2(\Omega) - \lambda_2(\Theta))^{\frac{1}{d+1}}\Big].\end{aligned}$$